STABILITY OF DEFORMATION OF VISCOELASTIC-PLASTIC BODIES

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There have been many studies of the stability of plastically deformable media. Specific problems are solved in [1, 2], etc. In these studies it has been assumed that the process of loss of stability can be investigated in the quasi-static formulation, i.e., an attempt is made to find the values of the external loads at which, together with the unperturbed equilibrium mode, the adjacent perturbed equilibrium state is possible, the transition from the unperturbed to the adjacent perturbed state being assumed to take place without unloading.

The results thus obtained are in agreement with general experimental concepts.

Below it is shown that the use of the model of a viscoelastic-plastic hardening body leads to a process of stability loss in which the material is plastically deformed, which justifies the use of the tangent-modulus formulation.

It is established that if the external loads are conservative, then for viscoelastic-plastic bodies loss of stability will occur in the static instability mode.

The stability of systems under creep conditions was previously examined in [3–8].

1. Let us consider the viscoelastic-plastic body whose mechanical model is shown in Fig. 1. The relation between the states of strain and stress in this body are determined in conformity with [9].

The body remains elastic as long as

$$s_{ij}s_{ij} < k^2(0), \qquad s_{ij} = \sigma_{ij} - {}^1/_3\sigma_{kk}\delta_{ij}, \tag{1.1}$$

where

$$\sigma_{ij} = \lambda e_{kk}{}^e\delta_{ij} + 2\mu e_{ij}{}^e. \tag{1.2}$$

If $s_{ij}s_{ij} \geq k^2(\varkappa)$, then

$$e_{ij} = e_{ij}{}^e + e_{ij}{}^p, \qquad \varkappa = e_{ij}{}^p e_{ij}{}^p. \tag{1.3}$$

In this case the elastic strains $e_{ij}{}^e$ are related to the stresses by Hooke's law (1.2). The plastic strain rates are

$$\dot{e}_{ij}^{p} = 0, \quad \text{if} \quad (s_{ij} - ce_{ij}{}^p)(s_{ij} - ce_{ij}{}^p) < k^2(\varkappa),$$

$$\dot{e}_{ij}^{p} = \psi(s_{ij} - ce_{ij}{}^p - \eta\dot{e}_{ij}^{p}),$$

$$\text{if} \quad (s_{ij} - ce_{ij}{}^p - \eta\dot{e}_{ij}^{p})(s_{ij} - ce_{ij}{}^p - \eta\dot{e}_{ij}^{p}) = k^2(\varkappa). \tag{1.4}$$

The total strains are related with the displacements by the Cauchy relations

$$e_{ij} = {}^1/_2(u_{i,j} + u_{j,i}).$$

To relations (1.2)–(1.5) it is necessary to add the equilibrium equations [10]

$$[\sigma_{jk}(\delta_{ik} + u_{i,k})]_{,j} + X_i = 0. \tag{1.6}$$

Let the surface forces p_i be given on the part S_F of the surface of an elastic-plastic body and the displacements u_i on the part S_V, the quantities p_i and u_i tending to or assuming the time-independent values $\overset{\circ}{p_i}$ and $\overset{\circ}{u_i}$ with increase in time.

Let the solution of the system of equations (1.1)–(1.6) with these boundary conditions be $\sigma_{ij}{}^\circ(x_k, t)$, $e_{ij}{}^\circ(x_k, t)$, $e_{ij}{}^{p\circ}(x_k, t)$, $u_i{}^\circ(x_k, t)$. We assume that as time increases these solutions tend to $\sigma_{ij}{}^\circ(x_k)$, $e_{ij}{}^\circ(x_k)$, $e_{ij}{}^{p\circ}(x_k)$, $u_i{}^\circ(x_k)$.

In what follows we will investigate the stability of this process with respect to small perturbations of the boundary conditions, mass forces, and deviations of the configuration of the body from the given geometric dimensions.

We find the solution for the perturbed motion in the form:

$$\sigma_{ij} = \sigma_{ij}^\circ(x_k, t) + \sigma_{ij}{}^+, \quad e_{ij} = e_{ij}{}^\circ(x_k, t) + e_{ij}{}^+,$$

$$e_{ij}{}^p = e_{ij}^{p\circ}(x_k, t) + e_{ij}^{p+}, \; u_i = u_i{}^\circ(x_k, t) + u_i{}^+. \tag{1.7}$$

It is assumed that the stability of motion can be judged from the linearized system of equations which we obtain by assuming that the components with a plus sign are small and retaining only linear terms of the expansion.

In the plastic region

$$e_{ij}{}^+ = e_{ij}^{e+} + e_{ij}^{p+}, \qquad \sigma_{ij}{}^+ = \lambda e_{kk}^{e+}\delta_{ij} + 2\mu e_{ij}^{e+},$$

$$(s_{ij}{}^\circ - ce_{ij}^{p\circ} - \eta\dot{e}_{ij}^{p\circ})(s_{ij}{}^+ - ce_{ij}^{p+} - \eta\dot{e}_{ij}^{p+}) =$$

$$= 2k(\varkappa_0)\,\partial k/\partial\varkappa\,|_{\varkappa=\varkappa_0}\, e_{ij}^{p\circ}e_{ij}^{p+},$$

$$\dot{e}_{ij}^{p+} = \psi^+(s_{ij}{}^\circ - ce_{ij}^{p\circ} - \eta\dot{e}_{ij}^{p\circ}) +$$

$$+ \psi^\circ(s_{ij}{}^+ - ce_{ij}^{p+} - \eta\dot{e}_{ij}^{p+}). \tag{1.8}$$

For $e_{ij}{}^+$ we obtain

$$e_{ij}{}^+ = {}^1/_2(u_{i,j}^+ + u_{j,i}^+). \tag{1.9}$$

The linearized equilibrium equations and boundary conditions on S_F have the form [10]:

$$\sigma_{ij,j}^+ + (\sigma_{jk}{}^\circ u_{i,k}^+)_{,j} + X_i{}^+ - \rho\ddot{u}_i^+ = 0,$$

$$\sigma_{ij}{}^+ n_j + \sigma_{jk}{}^\circ u_{i,k}^+ n_j = p_i{}^+. \tag{1.10}$$

Relations (1.10) were obtained for an elastic body, but since they have a geometric significance unrelated to the properties of the body they can also be used for inelastic bodies.

Eliminating the quantities $e_{ij}{}^{e+}$ and $e_{ij}{}^{p+}$ from relations (1.8), (1.9), we obtain

$$\lambda\dot{u}_{k,k}^+\delta_{ij} + \mu(\dot{u}_{i,j}^+ + \dot{u}_{i,k}^+) - \dot{\sigma}_{ij}^+ =$$

$$= \psi^\circ[2\mu\sigma_{ij}{}^+ - {}^2/_3\mu(3\lambda + 2\mu)u_{k,k}^+\delta_{ij} -$$

$$- c\lambda u_{k,k}^+\delta_{ij} - c\mu(u_{i,j}^+ + u_{j,i}^+) + c\sigma_{ij}{}^+ - \lambda\eta\dot{u}_{k,k}^+\delta_{ij} -$$

$$- \mu\eta(\dot{u}_{i,j}^+ + \dot{u}_{j,i}^+) + \eta\dot{\sigma}_{ij}^+] + k^{-2}[\lambda\dot{u}_{m,m}^+\delta_{kl} +$$

$$+\mu(u^{+}_{k,l}+u^{+}_{l,k})-\sigma^{+}_{lk}](s^{\circ}_{kl}-ce^{p\circ}_{kl}-$$
$$-\eta e^{\cdot p\circ}_{kl})(s_{ij}^{\circ}-ce^{p\circ}_{ij}-\eta e^{\cdot p\circ}_{ij}), \qquad (1.11)$$

$$[2\mu\sigma_{ij}^{+}-{}^{2}/_{3}\mu(3\lambda+2\mu)u^{+}_{k,k}\delta_{ij}-$$
$$-c\lambda u^{+}_{k,k}\delta_{ij}-c\mu(u^{+}_{i,j}+u^{+}_{j,i})+c\sigma_{ij}^{+}-$$
$$-\lambda\eta u^{\cdot+}_{k,k}\delta_{ij}-\mu\eta(u^{\cdot+}_{i,j}+u^{\cdot+}_{j,i})+$$
$$+\eta\sigma^{\cdot+}_{ij}](s_{ij}^{\circ}-ce^{p\circ}_{ij}-\eta e^{\cdot p\circ}_{ij})=$$
$$=k_1e^{p\circ}_{ij}[\lambda u^{+}_{k,k}\delta_{ij}+\mu(u^{+}_{i,j}+u^{+}_{j,i})-\sigma_{ij}^{+}],$$
$$k_1=2k(\varkappa_0)\,\partial k/\partial\varkappa|_{\varkappa=\varkappa_0}. \qquad (1.12)$$

The unperturbed equilibrium will be stable or unstable depending on the behavior of the perturbations as t increases without bound.

2. We find the solution of Eqs. (1.10)–(1.12) in the form:

$$u_i^{+}(x_k,t)=f_n(t)\varphi_{in}(x_k),$$
$$\sigma_{ij}^{+}(x_k,t)=\psi_m(t)\sigma_{ijm}(x_k). \qquad (2.1)$$

Here, $f_n(t)$ and $\psi_m(t)$ are certain functions of time, $\varphi_{in}(x_k)$ are the eigenvectors of the elastic problem, and

$$\sigma_{ijm}(x_k)=\lambda\varphi_{nm,n}\delta_{ij}+2\mu(\varphi_{im,j}+\varphi_{jm,i}). \qquad (2.2)$$

We rewrite the equation of Galerkin's method [10] in the form:

$$\int_V[\sigma^{+}_{ij,j}+X_i^{+}-\rho u^{\cdot\cdot+}_{i}]\varphi_{im}dV-\int_V\sigma^{\circ}_{jk}u^{+}_{i,k}\varphi_{im,j}dV-$$
$$-\iint_{S_F}(\sigma_{ij}^{+}n_j-p_i^{+})\varphi_{im}dS=0$$
$$(m=1,2,3,\ldots,\ldots). \qquad (2.3)$$

Substituting expression (2.1) into (2.3) we obtain the system of equations

$$d^2f_m/dt^2+b_{mn}f_n-a_{mn}\psi_n=0$$
$$(m=1,2,3,\ldots,\ldots), \qquad (2.4)$$
$$a_{mn}=\iint_{S_F}\sigma_{ijn}n_j\varphi_{im}dS-\int_V\sigma_{ijn,j}\varphi_{im}dV,$$
$$b_{mn}=\iint_{S_F}p_i^{+}(\varphi_{in})\varphi_{im}dS+\int_V X_i^{+}(\varphi_{in})\varphi_{im}dV-$$
$$-\int_V\sigma^{\circ}_{jk}\varphi_{in,k}\varphi_{im,j}dV. \qquad (2.5)$$

Substituting (2.1) into (1.11) after multiplying by σ_{ijn} and integrating over the volume V, we obtain

$$A_{mn}\frac{df_m}{dt}+B_{mn}f_m-C_{mn}\frac{d\psi_m}{dt}+D_{mn}\psi_m=0$$
$$(n=1,2,3,\ldots,\ldots), \qquad (2.6)$$
$$A_{mn}=\int_V[a\lambda\varphi_{km,k}\delta_{ij}+a\mu(\varphi_{im,j}+\varphi_{jm,i})-$$
$$-k^{-2}f_{kl}^{\circ}f_{ij}^{\circ}(\lambda\varphi_{km,k}\delta_{kl}+\mu(\varphi_{km,l}+\varphi_{lm,k}))]\sigma_{ijn}dV,$$
$$B_{mn}=\int_V[a_1\varphi_{km,k}\delta_{ij}+a_2(\varphi_{im,j}+\varphi_{jm,i})]\sigma_{ijn}dV,$$
$$f_{ij}^{\circ}=(s_{ij}^{\circ}-ce^{p\circ}_{ij}-\eta e^{\cdot p\circ}_{ij}),$$
$$C_{mn}=\int_V[a\sigma_{ijm}+k^{-2}\sigma_{klm}f_{kl}^{\circ}f_{ij}^{\circ}]\sigma_{ijn}dV,$$
$$D_{mn}=\int_V a_3\sigma_{ijm}\sigma_{ijn}dV, \qquad (2.7)$$
$$a=1+2\eta\psi^{\circ},\quad a_1=2\psi^{\circ}[{}^{2}/_{3}\mu(3\lambda+2\mu)+c\lambda],$$
$$a_2=2c\mu\psi^{\circ},\quad a_3=2\psi^{\circ}(2\mu-c). \qquad (2.8)$$

Relation (1.12) reduces to the form

$$A_ndf_n/dt+B_nf_n-C_nd\psi_n/dt-D_n\psi_n=0$$
$$(n=1,2,3,\ldots), \qquad (2.9)$$
$$A_n=\int_V f_{ij}^{\circ}[\lambda\eta\varphi_{kn,k}\delta_{ij}+\mu\eta(\varphi_{in,j}+\varphi_{jn,i})]dV,$$
$$B_n=\int_V\Big[f_{ij}^{\circ}\frac{2(3\lambda+2\mu)}{3\mu}\varphi_{kn,k}\delta_{ij}+$$
$$+(cf_{ij}^{\circ}+k_1e^{p\circ}_{ij})(\lambda\varphi_{kn,k}\delta_{ij}+\mu(\varphi_{in,j}+\varphi_{jn,i}))\Big]dV,$$
$$C_n=\eta\int_V f_{ij}^{\circ}\sigma_{ijn}dV,$$
$$D_n=\int_V(2\mu+c+k_1)(f_{ij}^{\circ}+e_{ij}^{\circ p})\sigma_{ijn}dV. \qquad (2.10)$$

The system of linear differential equations (2.4), (2.6), and (2.9) can be reduced to the normal form

$$dz/dt=D(t)\,z, \qquad (2.11)$$

where $z=(f_1,f_2,\ldots,f_m,\ldots;\psi_1,\psi_2,\ldots,\psi_m,\ldots;\theta_1,\theta_2,\ldots,\theta_m,\ldots)$ is a vector and D some operator.

The system of equations (2.4), (2.6), (2.9) can be represented in the form $Adz/dt+Bz=0$, where A and B are infinite matrices

$$A=\begin{Vmatrix}A_{ij} & C_{ij} & 0\\ I & 0 & 0\\ 0 & 0 & I\end{Vmatrix},\qquad B=\begin{Vmatrix}B_{ij} & D_{ij} & 0\\ 0 & 0 & -I\\ b_{ij} & a_{ij} & 0\end{Vmatrix}. \qquad (2.12)$$

Here, A_{ij}, B_{ij}, C_{ij}, D_{ij}, a_{ij}, and b_{ij} $(i;j=1,2,3,\ldots)$ are infinite matrices whose coefficients are determined by (2.5), (2.7), and (2.9), I is the infinite identity matrix, and 0 the infinite null matrix.

Let R be some operator:

$$R=\begin{bmatrix}0 & I & 0\\ B_{ij}^{-1} & B_{ij}^{-1}A_{ij} & 0\\ 0 & 0 & I\end{bmatrix}.$$

We assume that the matrix B_{ij} is invertible. The operator R is the inverse of matrix A, since AR = RA = I. Then equation $Adz/dt+Bz=0$ can be written in the form

$$dz/dt+A^{-1}Bz=0.$$

Thus, system (2.4), (2.6), (2.9) reduces to the normal form (2.11), where

$$D=-\begin{bmatrix}0 & 0 & I\\ B_{ij}^{-1}C_{ij} & B_{ij}^{-1}D_{ij} & B_{ij}^{-1}A_{ij}\\ a_{ij} & b_{ij} & 0\end{bmatrix}.$$

3. Let us consider Eq. (2.11). It is assumed that

$$\| D(t, \beta) - C(\beta) \| \to 0 \text{ as } t \to \infty. \tag{3.1}$$

The differential equation (2.11) goes over into the differential equation with constant coefficients

$$dz/dt = C(\beta) z. \tag{3.2}$$

Let the spectrum of the operator C lie in the left half-plane. Then for the operator e^{Ct} we have the estimate ([11], p. 20)

$$\| e^{ct} \| \leqslant N e^{-\nu t} \qquad (\nu > 0). \tag{3.3}$$

Together with Eqs. (2.11) and (3.2), we consider the system of differential equations

$$dz/dt = [D(t, \beta) + \delta I] z, \quad dz/dt = [C(\beta) + \delta I] z. \tag{3.4}$$

Here, δ is so small that the spectrum of the operator $C + \delta I$ lies in the left half-plane, and for the operator $e^{(C+\delta I)}$ there is an estimate

Fig. 1

of type (3.3). Then there exists a bounded quadratic form

$$W(z) = \int_0^\infty (e^{(C+\delta I)t} z, \qquad e^{(C+\delta I)t} z)\, dt. \tag{3.5}$$

In fact,

$$W(z) = \int_0^\infty \| e^{(c+\delta I)t} z \|^2 dt \leqslant N_1 \| z \|^2 \int_0^\infty e^{-\nu_1 t} dt < \infty,$$

i.e.,

$$| W(z) | \leqslant \left(N_1 \int_0^\infty e^{-\nu_1 t} dt \right) \| z \|^2 = N_0 \| z \|^2, \tag{3.6}$$

where N_0 and N_1° are certain positive constants.

In view of system (3.4) the total derivative of this function W(z) will be

$$\Gamma(z) [C + \delta I] z = -\| z \|^2. \tag{3.7}$$

And this, in view of Lyapunov's general theorem of stability [12, 13], compels us to conclude that the function W given by Eq. (3.5) is a Lyapunov function in the sense of the stationary equation (3.4).

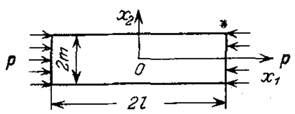

Fig. 2

We will find the total derivative with respect to t of the function W. In view of system (3.4),

$$\Gamma(z) [D(t, \beta) + \delta I] z =$$

$$= 2 \int_0^\infty (e^{(C+\delta I)s} z, e^{(C+\delta I)s} D(t, \beta) z)\, ds +$$

$$+ 2\delta \int_0^\infty (e^{(C+\delta I)s} z, e^{(C+\delta I)s} z)\, ds. \tag{3.8}$$

Since

$$2\delta \int_0^\infty (e^{(C+\delta I)t} z, e^{(C+\delta I)t} z)\, dt =$$

$$= \| z \|^2 - 2 \int_0^\infty (e^{(C+\delta I)t} z, e^{(C+\delta I)t} C(\beta) z)\, dt,$$

we rewrite (3.8) in the form

$$\Gamma(z) [D(t, \beta) + \delta I] z = -\| z \|^2 + 2 \int_0^\infty (e^{(C+\delta I)s} z,$$

$$e^{(C+\delta I)s} [D(t, \beta) - C(\beta)] z)\, ds. \tag{3.9}$$

Since W is a quadratic form with constant coefficients, there is a nonzero number $\varepsilon(\beta)$ such that when the inequality

$$\| D(t, \beta) - C(\beta) \| < \varepsilon$$

is satisfied the right side of the last relation is a positive definite quadratic form of the variable z.

Since $D(t, \beta)$ tends to the limit $C(\beta)$ as t increases without bound, for any ε, however small, there is a T such that at $t \geq T$ the absolute value of the differences $D(t, \beta) - C(\beta)$ will be smaller than ε and, consequently, for all values of t exceeding T the derivative $\Gamma(z)[D(t, \beta) + \delta I] z$ will be a negative function.

Thus, we have the following theorem. If the spectrum of matrix C lies in the left half-plane, the unperturbed motion of the nonstationary system is asymptotically stable in the Lyapunov sense.

Hence we may conclude that as $t \to \infty$ the solutions of Eqs. (2.11) behave approximately in the same way as the solutions of Eqs. (2.3).

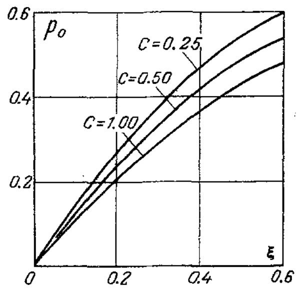

Fig. 3

4. The theorem proved makes it possible to simplify our investigation of the behavior of the solution of system of equations (1.10)–(1.12).

If in the coefficients of Eqs. (1.10)–(1.12) we let t tend to infinity, then in the limit we obtain a system of equations with stationary coefficients.

Finding the solution of the limiting system of equations in the form (2.1) by the method developed in §2, we reduce the solution of this system to the system of ordinary differential equations

$$dz/dt = C(\beta) z, \tag{4.1}$$

the operators $D(t, \beta)$ in (2.11) and $C(\beta)$ in (4.1) possessing the properties (3.1). Thus, the stability of the solution of system (1.10)–(1.12) can be investigated with respect to the limiting system of equations on the assumption that series (2.1) converge.

Letting the time t in (1.10)–(1.12) go to infinity, we obtain the limiting system of equations in the form

$$\sigma_{ij,j}^{+} + (\sigma_{jk}^{\circ} u_{i,k}^{+})_{,j} + X_i^{+} - \rho \ddot{u}_i^{+} = 0,$$

$$\lambda \dot{u}_{m,m}^{+} \delta_{ij} + \mu (\dot{u}_{i,j}^{+} + \dot{u}_{j,i}^{+}) - \dot{\sigma}_{ij}^{+} = k^{-2} [\lambda \dot{u}_{n,n}^{+} \delta_{kl} +$$

$$+ \mu (\dot{u}_{k,l}^{+} + \dot{u}_{l,k}^{+}) - \dot{\sigma}_{kl}^{+}] (s_{kl}^{\circ} - c e_{kl}^{p\circ}) (s_{ij}^{\circ} - c e_{ij}^{p\circ}),$$

$$\{(2\mu + c) \dot{\sigma}_{ij}^{+} - [{}^2/_3 \mu (3\lambda + 2\mu) +$$

$$+ c\lambda] u_{k,k}^{+}\delta_{ij} - c\mu (u_{i,j}^{+} + u_{j,i}^{+}) -$$

$$- \lambda\eta u_{k,k}^{\cdot +}\delta_{ij} - \mu\eta (u_{i,j}^{\cdot +} + u_{j,i}^{\cdot +}) + \eta\sigma_{ij}^{\cdot +}\} (s_{ij}^{\circ} - ce_{ij}^{p\circ}) =$$

$$= k_1 e_{ij}^{p\circ}[\lambda u_{k,k}^{+}\delta_{ij} + \mu (u_{i,j}^{+} + u_{j,i}^{+}) - \sigma_{ij}^{+}], \qquad (4.2)$$

with boundary conditions at the surface S_F

$$\sigma_{ij}^{+}n_j + \sigma_{jk}^{\circ}u_{i,k}^{+}n_j = p_i^{+}. \qquad (4.3)$$

In investigating the boundary value problem (4.3) for Eqs. (4.2) we will use the method described in [10]. The solution of these equations is found in the form

$$u_j^{+}(x_k, t) = U_j(x_k) e^{i\omega t},$$

$$\sigma_{jk}^{+}(x_l, t) = \sigma_{jk}(x_l) e^{i\omega t}. \qquad (4.4)$$

Substituting (4.4) into (4.2), we obtain

$$\sigma_{ij,i} + (\sigma_{jk}^{\circ}U_{i,k})_{,i} + X_i(U_k; \omega) + \rho\omega^2 U_i = 0,$$

$$\sigma_{ij}n_j + \sigma_{jk}^{\circ}U_{i,k}n_j = p_i(U_k; \omega),$$

$$\lambda U_{m,m}\delta_{ij} + \mu (U_{j,i} + U_{j,i}) - \sigma_{ij} = k^{-2}[\lambda U_{n,n}\delta_{ki} +$$

$$+ \mu (U_{k,l} + U_{l,k}) - \sigma_{kl}] (s_{kl}^{\circ} - ce_{kl}^{p\circ})(s_{ij}^{\circ} - ce_{ij}^{p\circ}), \qquad (4.5)$$

$$\Big\{(2\mu + c + s\eta)\sigma_{ij} - \Big[\frac{(3\lambda + 2\mu)}{3\mu} + c\lambda + s\lambda\eta\Big] U_{k,k}\delta_{ij} -$$

$$- (c\mu + s\mu\eta)(U_{i,j} + U_{j,i})\Big\} (s_{ij}^{\circ} - ce_{ij}^{p\circ}) =$$

$$= k_1 e_{ij}^{p\circ}[\lambda U_{k,k}\delta_{ij} + \mu (U_{i,j} + U_{j,i}) - \sigma_{ij}], \quad s = i\omega. \quad (4.6)$$

From (4.6) we can obtain

$$\sigma_{ij}(x_l) = \lambda U_{k,k}\delta_{ij} + \mu(U_{i,j} + U_{j,i}) +$$

$$+ \frac{2\mu^2 [{}^2/_3 U_{k,k}\delta_{lm} - (U_{l,m} + U_{m,l})] (s_{lm}^{\circ} - ce_{lm}^{p\circ})}{k^2 [2\mu + c + s\eta + k_1 k^{-2} e_{lm}^{p\circ} (s_{lm}^{\circ} - ce_{lm}^{p\circ})]} \times$$

$$\times (s_{ij}^{\circ} - ce_{ij}^{p\circ}). \qquad (4.7)$$

Equations (4.7) may be treated as the relation between the states of stress and strain in an anisotropic elastic body with complex modulus of elasticity.

By repeating the arguments of [10] it can be shown that if the external forces possess a potential, loss of stability can occur only in the static instability mode, since in this case the corresponding problem is self-adjoint.

5. As an example we will investigate the stability of the process of deformation of a rectangular infinite strip compressed by a uniform pressure p applied on two opposite sides (Fig. 2).

We will consider the case of plane strain, i.e., loss of stability occurs in the plane x_1Ox_2. Neglecting, for simplicity, compressibility and assuming that k = const, we write relation (4.7) in the form:

$$\sigma_{ij} - \sigma\delta_{ij} = \mu (U_{i,j} + U_{j,i}) -$$

$$- \frac{2\sqrt{2}\mu^2}{k(2\mu + c)} (U_{2,2} - U_{1,1}) (s_{ij}^{\circ} - ce_{ij}^{p\circ}), \quad \sigma = {}^1/_3\sigma_{kk}. \qquad (5.1)$$

The state of stress in the strip up to the loss of stability is determined by the expressions

$$\sigma_{11}^{\circ} = -p, \quad \sigma_{22}^{\circ} = 0, \quad \sigma_{12}^{\circ} = 0,$$

$$e_{11}^{p\circ} = -e_{22}^{p\circ} = (k\sqrt{2} - p)/2c, \quad e_{12}^{p\circ} = 0. \qquad (5.2)$$

The equilibrium equations, determined from (4.5) using (5.2), are written as follows:

$$\sigma_{11,1} + \sigma_{12,2} - pU_{1,11} = 0,$$
$$\sigma_{12,1} + \sigma_{22,2} - pU_{2,22} = 0. \qquad (5.3)$$

The boundary conditions have the form

$$\sigma_{22} = 0, \ \sigma_{12} = 0 \quad \text{at} \quad x_1 = \pm l, \ x_2 = \pm m. \qquad (5.4)$$

Using equilibrium equations (5.3), the stress-strain relation (5.1), and the incompressibility condition, we obtain the starting system of equations

$$(2\gamma_0 + p_0 - 1)\ U_{1,112} - U_{1,222} - (1 + p_0)\ U_{2,111} +$$

$$+ (1 - 2\gamma_0)\ U_{2,122} = 0$$

$$U_{1,1} + U_{2,2} = 0, \quad p/\mu = p_0,$$

$$c/\mu = c_0, \quad 8/(2 + c_0) = \gamma_0. \qquad (5.5)$$

The solution of Eqs. (5.5) should be found in the form

$$U_1 = f_1(x_2) \sin \alpha x_1, \qquad U_2 = f_2(x_2) \cos \alpha x_1. \qquad (5.6)$$

After substituting (5.6) in (5.5) we obtain an equation for the function $f_2(x_2)$:

$$d^4f_2/dx_2^4 - \alpha^2 (2 - p_0 - \gamma_0)\ d^2f_2/dx_2^2 + \alpha^4 (1 - p_0)\ f_2 = 0. \qquad (5.7)$$

Equation (5.7) is easily integrated. The function $f_2(x_2)$ found from (5.7) is a sum of even and odd functions.

Confining our attention to the lateral buckling of a plate, we obtain

$$f_2(x_2) = A_1 \operatorname{ch} w_1x_2 + A_2 \operatorname{ch} w_2x_2. \qquad (5.8)$$

Here, A_i are arbitrary constants,

$$w_1 = \{({}^1/_2\alpha^2)\ \{(2 - \gamma_0 - p_0) + [(p_0 + \gamma_0)^2 - 4\gamma_0]^{1/2}\}\}^{1/2},$$

$$w_2 = \{({}^1/_2\alpha^2)\ \{(2 - \gamma_0 - p_0) - [(p_0 + \gamma_0)^2 - 4\gamma_0]^{1/2}\}\}^{1/2}.$$

After determining $f_1(x_2)$ and $f_2(x_2)$ we find U_1 and U_2. Using relations (5.1) and equilibrium equations (5.3), we obtain the stress components of the perturbed state, whose substitution into the boundary conditions (5.4) at the free surface leads to the consideration of a system of algebraic linear homogeneous equations in the constants of integration.

In the event of loss of stability this system has a nontrivial solution, i.e., its determinant is equal to zero. Hence for determining the critical pressure we obtain the equation

$$\frac{w_2(\alpha^2 - w_2^2)[w_1^2 - \alpha^2(1 - p_0)]}{w_1(\alpha^2 - w_1^2)[w_2^2 - \alpha^2(1 - p_0)]} = \operatorname{th}(mw_2)\operatorname{cth}(mw_1). \qquad (5.9)$$

In the same way (only the root indices of the trigonometric coefficients change) we can obtain an equation for determining the critical load, assuming bilateral bulging in compression or necking in tension.

The boundary conditions will be satisfied if $\alpha = n\pi/l$ (n is the number of half-waves). For thin strips Eq. (5.9) can be simplified, since it is possible to use a power series expansion of the trigonometric coefficient, limited, in view of the smallness of the thickness as compared with the length, to the second order:

$$p_0^3 - [(1/\alpha^2m^2) - \gamma_0]\ p_0^2 + \gamma_0 (2 + \alpha^2m^2\gamma_0)\ p_0 - (\alpha m\gamma_0)^2 = 0. \qquad (5.10)$$

Figure 3 is a graph of the critical load versus $(\alpha m)^2 = \xi$ for Eq. (5.10).

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